Day 1: The Probabilistic Method

Exercise 1. (Russia 1999)

In a certain school, every boy likes at least one girl. Prove that we can find a set S of at least half the students in the school such that each boy in S likes an odd number of girls in S.

Walkthrough:

(a) Flip a coin for every girl to determine whether she goes in S or not. What is the expected number of girls in S?
(b) Put every boy who likes an odd number of girls in S into S. What is the expected number of boys in S?
(c) What is the expected size of S? Conclude.

Exercise 2. Show that any graph G with m edges has a bipartite subgraph with $\geq \frac{m}{2}$ edges.

Walkthrough:

(a) Flip a coin on every vertex and define a corresponding bipartite subgraph.
(b) Show that the expected number of edges in the subgraph is $\frac{m}{2}$, and conclude.

Exercise 3. (USAMO 2012, problem 2)

A circle is divided into 432 congruent arcs by 432 points. The points are colored in four colors such that some 108 points are colored Red, some 108 points are colored Green, some 108 points are colored Blue, and the remaining 108 points are colored Yellow. Prove that one can choose three points of each color in such a way that the four triangles formed by the chosen points of the same color are congruent.

Walkthrough:

(a) Consider a random symmetry of the 432-gon formed by the points. How many are there? (Don't include the identity.)
(b) Color the red points that land on green points orange. What's the expected number of orange points? How many orange points are guaranteed to achievable?
(c) Modify and repeat step (b).
(d) Conclude.

Exercise 4. (Erdős)

Let $R(s)$ denote the Ramsey number of s, i.e., the smallest integer n for which, when one colors the edges of K_n either red or blue, there must be a monochromatic K_s.

Show that $R(s) > 2^{s/2}$ for $s \geq 3$.

Walkthrough:

(a) Let $n = \lfloor 2^{s/2} \rfloor$. Showing that $R(s) > n$ is showing that there existing a coloring of K_n with no monochromatic K_s. Randomly color each edge of K_n red or blue. What is the probability that a given set of s vertices forms a monochromatic K_s?
(b) Show that it suffices to show $\binom{n}{s} < 2^{\binom{s}{2}-1}$, and verify this is true by showing that $\binom{n}{s} < \frac{n^s}{2^s} < 2^{\binom{s}{2}-1}$.

Bonus Homework 5. Alice marks ten points in the plane. Is it always possible for Bob to place ten unit discs to cover all ten points, so that no two discs overlap?

Theorem 6 (Markov's Inequality). *If $X \geq 0$ is a random variable and $a > 0$,*

$$P(X \geq a) \leq \frac{1}{a} \mathrm{E}[X],$$

with equality iff $X \in \{0, a\}$.

Homework 7. Prove this.

Definition 8. The covariance of two variables X and Y is

$$\mathrm{Cov}(X, Y) := \mathrm{E}[(X - \mathrm{E}[X])(Y - \mathrm{E}[Y])]$$

Homework 9. Show that $\mathrm{Cov}(X, Y)$ can also be written as $\mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]$.

Definition 10. The variance of a random variable X is

$$\mathrm{Var}(X) := \mathrm{Cov}(X, X) = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] = \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2.$$

Theorem 11 (Chebyshev's Inequality). *If X is a random variable and $a > 0$,*

$$P(|X - \mathrm{E}[X]| \geq a) \leq \frac{\mathrm{Var}(X)}{a^2},$$

with equality iff $X - \mathrm{E}[X] \in \{0, a, -a\}$.

Proof. This is the direct result of plugging in $(X - \mathrm{E}[X])^2$ for X and a^2 for a into Markov's Inequality (**6**). □

Exercise 12. (USAMO 2012, problem 6)

For integer $n \geq 2$, let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$x_1 + x_2 + \ldots + x_n = 0, \qquad \text{and} \qquad x_1^2 + x_2^2 + \ldots + x_n^2 = 1.$$

For each subset $A \subseteq \{1, 2, \ldots, n\}$, define

$$S_A = \sum_{i \in A} x_i.$$

(If A is the empty set, then $S_A = 0$.)

Prove that for any positive number λ, the number of sets A satisfying $S_A \geq \lambda$ is at most $2^{n-3}/\lambda^2$. For which choices of $x_1, x_2, \ldots, x_n, \lambda$ does equality hold?

Walkthrough:

(a) Flip n coins and let $X_i = x_i$ if the i^{th} coin comes up heads, and $X_i = 0$ if it comes up tails. Let A be the set of indices of coins that came up heads. Write

$$\sum_{i=1}^{n} X_i = S_A.$$

(b) What is $\mathrm{E}[X_i]$? $\mathrm{E}[S_A]$? $\mathrm{Var}(X_i)$? $\mathrm{Var}(S_A)$?

(c) What does Chebyshev's inequality (**11**) say when you plug in 2λ?

(d) Show that $P(S_A \geq \lambda) = P(-S_A \geq \lambda)$.

(e) Conclude that the inequality given in the problem holds.

(f) What does the equality case of Chebyshev's inequality say about the equality case for this problem? Conclude.

Day 2: Analytic Number Theory

Definition 13. For functions f and g where there exists constants N and C such that $|g(n)| \le Cf(n)$ for all $n > N$, we can write $g = O(f)$. In other words, $g = O(f)$ if and only if $f(n)$ is positive for sufficiently large n and $\limsup_{n\to\infty} \left|\frac{g(n)}{f(n)}\right| < \infty$.

Definition 14. For functions f and g, if for every $\epsilon > 0$ there exists some N such that $|g(n)| \le \epsilon f(n)$ for all $n > N$, we can write $g = o(f)$. In other words, $g = o(f)$ if and only if $f(n)$ is positive for sufficiently large n and $\lim_{n\to\infty} \left|\frac{g(n)}{f(n)}\right| = 0$.

Definition 15. For functions f and g, if for every $\epsilon > 0$ there exists some N such that $\max\left(\frac{f(n)}{g(n)}, \frac{g(n)}{f(n)}\right) < 1+\epsilon$, we can write $g \sim f$. In other words, $g \sim f$. if and only if $\lim\limits_{n\to\infty} \frac{g(n)}{f(n)} = 1$.

(If you don't understand the limit definitions, don't worry about it.)

Definition 16. The *prime counting function* $\pi(n)$ is the number of primes less than or equal to n.

The prime number theorem states that $\pi(n) \sim \frac{n}{\ln n}$. We prove the following weaker statement.

Theorem 17 (Chebyshev)**.**

$$\pi(n) = O\left(\frac{n}{\ln n}\right)$$

Walkthrough: (Variant of Erdős).

(a) Show that if $n < p \le 2n$, $p \,\Big|\, \binom{2n}{n}$. Conclude that $n^{\pi(2n)-\pi(n)} < 4^n$, and that $\pi\left(2^k\right) - \pi\left(2^{k-1}\right) < \frac{2^k}{k-1}$.

(b) Show that $\pi(4^m) - 1 < \sum\limits_{k=2}^{2m} \frac{2^k}{k-1}$, and conclude that $\pi(4^m) < 2^{m+1} + \frac{2^{2m+1}}{m}$.

(c) Conclude.

Definition 18. The *von Mangoldt function* is

$$\Lambda(n) := \begin{cases} \ln p & \text{if } n > 1 \text{ is a power of } p \\ 0 & \text{otherwise.} \end{cases}$$

Homework 19. Show the following key property of this function:

$$\sum_{d|n} \Lambda(d) = \ln n.$$

Definition 20. The *second Chebyshev function* is

$$\psi(x) := \sum_{n\le x} \Lambda(n).$$

The prime number theorem is equivalent to $\psi(x) \sim x$. We prove the following weaker statement.

Theorem 21.

$$\psi(x) = O(x).$$

Proof. By Chebyshev (**17**),

$$\psi(x) = \sum_{n \le x} \Lambda(n) = \sum_{p \le x} \sum_{p^a \le x} \ln p = \sum_{p} \ln\left(p^{\lfloor \log_p(x) \rfloor}\right) \le \sum_{p} \ln(x) = (\ln x) \sum_{p} 1 = (\ln x)\pi(x) = O(x).$$

□

We now introduce a very useful tool in analytic number theory. Be warned that the following discussion does involve some calculus.

Theorem 22 (Abel summation formula)**.** *Given* $(a_n)_{n=1}^{\infty}$, *let*

$$A(x) = \sum_{n \le x} a_n.$$

If f *is a continuous function for* $x \ge 1$,

$$\sum_{n \le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\,\mathrm{d}t.$$

You can think of this as a more discrete version of integration by parts.

Bonus Homework 23. Prove this.

Theorem 24 (Weak version of Stirling's formula)**.**

$$\ln x! = x \ln x - x + O(\ln x)$$

Proof. Using $a_n = 1$, $A(x) = \lfloor x \rfloor$ and $f(x) = \ln(x)$ in the Abel Summation formula (**22**),

$$\begin{aligned}
\ln x! = \sum_{n \le x} \ln n = x \ln x - \int_1^x \frac{\lfloor x \rfloor}{x} = x \ln x - \int_1^x \frac{x - \{x\}}{x} \\
= x \ln x - \int_1^x 1 + \int_1^x \frac{\{x\}}{x} = x \ln x - (x-1) + O\left(\int_1^x \frac{1}{x}\right) \\
= x \ln x - x + O(\ln x)
\end{aligned}$$

□

Theorem 25.

$$\sum_{d \le x} \frac{\Lambda(d)}{d} = \ln x + O(1).$$

Proof. By homework **19** and theorems **21** and **24**,

$$x \ln x + o(x \ln x) = \sum_{n \le x} \ln n = \sum_{n \le x} \sum_{d \mid n} \Lambda(d) = \sum_{d \le x} \left\lfloor \frac{x}{d} \right\rfloor \Lambda(d) = \sum_{d \le x} x \frac{\Lambda(d)}{d} + O(\psi(x)) = x \sum_{d \le x} \frac{\Lambda(d)}{d} + O(x).$$

Dividing by x, we get the desired result. □

Homework 26. Show that

$$\sum_{n\in\mathbb{N}} \frac{\ln n}{n^2}$$

converges.

Theorem 27 (Merten's first theorem)**.**

$$\sum_{p\le x} \frac{\ln p}{p} = \ln x + O(1).$$

Proof. By Theorem **25** and Homework **26**,

$$\begin{aligned}
\ln x + O(1) &= \sum_{d\le x} \frac{\Lambda(d)}{d} = \sum_{p\le x} \sum_{p^a\le x} \frac{\ln p}{p^a} \\
&= \sum_{p\le x} \frac{\ln p}{p} + O\left(\sum_{p\le x} (\ln p)\left(\frac{1}{p^2} + \frac{1}{p^3} + \cdots\right)\right) \\
&= \sum_{p\le x} \frac{\ln p}{p} + O\left(\sum_{p\le x} \frac{2\ln p}{p^2}\right) = \sum_{p\le x} \frac{\ln p}{p} + O(1),
\end{aligned}$$

giving the desired result. □

Theorem 28 (Merten's second theorem)**.** *There exists a constant M such that*

$$\sum_{p\le x} \frac{1}{p} = \ln\ln x + M + O\left(\frac{1}{\ln x}\right).$$

Proof. Using $a_p = \frac{\ln p}{p}$ and $a_n = 0$ for n not prime, and $f(x) = \frac{1}{\ln x}$, in the Abel summation formula (**22**), we have that by Merten's first theorem (**27**),

$$\begin{aligned}
\sum_{p\le x} \frac{1}{p} &= \frac{A(x)}{\ln x} + \int_1^x \frac{A(t)}{t\ln^2 t}\,\mathrm{d}t = \frac{A(x)}{\ln x} + \int_2^x \frac{A(t)}{t\ln^2 t}\,\mathrm{d}t \\
&= \frac{\ln x + O(1)}{\ln x} + \int_2^x \frac{\ln t + O(1)}{t\ln^2 t}\,\mathrm{d}t \\
&= \left(1 + O\left(\frac{1}{\ln x}\right)\right) + \int_2^x \frac{1}{t\ln t}\,\mathrm{d}t + \int_2^x \frac{O(1)}{t\ln^2 t}\,\mathrm{d}t \\
&= \left(1 + O\left(\frac{1}{\ln x}\right)\right) + (\ln\ln x - \ln\ln 2) + \left(\int_2^\infty \frac{O(1)}{t\ln^2 t}\,\mathrm{d}t - \int_x^\infty \frac{O(1)}{t\ln^2 t}\,\mathrm{d}t\right) \\
&= \ln\ln x + \left(1 - \ln\ln 2 + \int_2^\infty \frac{O(1)}{t\ln^2 t}\,\mathrm{d}t\right) + O\left(\frac{1}{\ln x}\right),
\end{aligned}$$

as desired. □

M is known as the *Meissel-Mertens constant*, and has value approximately 0.2615.
