Magic Math

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BERKELEY
MATH CIRCLE

Far from Kansas (Graham-Pollak Theorem)

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Problem

For $n \in \mathbb{N}$, prove that it's impossible to partition a complete graph on $n+1$ vertices into less than n complete bipartite graphs.

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Suppose that 101 *players compete in several rounds of a game.*

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For two teams $\{a, b, c\}$ and $\{x, y, z\}$, the expression

$$(a+b+c)(x+y+z) = ax+ay+az+bx+by+bz+cx+cy+cz$$

simulates all pairs of people that face off against each other!

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simulates all pairs of people that face off against each other!

So if the players are $x_1, x_2, \ldots, x_{101}$ and every pair faces off once, terms such as these must sum to $\sum_{i<j} x_i x_j$.

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$$\sum_{round} \text{terms like } (a+b+c)(x+y+z) = \sum_{i<j} x_i x_j = \frac{1}{2} \sum_{i \neq j} x_i x_j$$

$$= \left(\sum x_i x_j - \sum_{i=j} x_i x_j \right) /2$$

$$= \left(\left(\sum x_i \right)^2 - \sum x_i^2 \right) /2$$

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Tricky function (2016 Brazil Olympic Revenge 5)

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Observations:

- $f(\mathbf{0}) = f(\mathbf{0} + \mathbf{0}) = f(\mathbf{0}) + f(\mathbf{0})$, so $\mathbf{0} \mapsto 0$

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Every element of T is a sum of multiples of $\mathbf{e}_i$!

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Prove that for any $\mathbf{x} \in T$, $\mathbf{x} \mapsto 0$.

$$\begin{aligned}(3,1,4,1,5,\dots) \mapsto\ & f((3,0,0,0,0,\dots)) + f((0,1,0,0,0,\dots)) \\ & + f((0,0,4,0,0,\dots)) + f((0,0,0,1,0\dots)) \\ & + f((0,0,0,0,5,\dots)) + \cdots \\ =\ & 0+0+0+0+0+\cdots = 0\end{aligned}$$

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$$

So aren't we done?

Tricky function (2016 Brazil Olympic Revenge 5)

$$\begin{aligned}
\lfloor 0.9 \rfloor &= 0 \\
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Wait no! $\lfloor 0.999\ldots \rfloor = \lfloor 1 \rfloor = 1$!

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Cannot induct infinitely! Only true if finitely many nonzero terms!

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$$f((1, 2, 4, 8, 16, \dots)) = f((0, 2, 4, 8, 16, \dots))$$

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$$f((1, 2, 4, 8, 16, \dots)) = f((0, 2, 4, 8, 16, \dots)) = 2f((0, 1, 2, 4, 8, \dots))$$

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This becomes $4f((0, 0, 1, 2, 4, \dots))$, $8f((0, 0, 0, 1, 2, \dots))$, etc!

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This becomes $4f((0,0,1,2,4,\dots))$, $8f((0,0,0,1,2,\dots))$, etc! But if $f(\mathbf{x})$ is divisible by every power of two, there's only one thing it can be...

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For any sequence $\mathbf{x}$ with increasing gcd, $f(\mathbf{x})$ must be divisible by larger and larger numbers, so must be 0.

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For any sequence $\mathbf{x}$ with increasing gcd, $f(\mathbf{x})$ must be divisible by larger and larger numbers, so must be 0. What about $\mathbf{x}$ in general?

$$\mathbf{x} = (x_1, x_2, x_3, \ldots)$$

We will write $\mathbf{x}$ as a sum of sequences we now know map to zero, by using the powers of 2 and 3.

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- *f is additive, so for any $\mathbf{a}, \mathbf{b} \in T$, $\mathbf{a} + \mathbf{b} \mapsto f(\mathbf{a}) + f(\mathbf{b})$,*
- *$\forall i \in \mathbb{N}$, $\mathbf{e}_i \mapsto 0$, where $\mathbf{e}_i$ has i^{th} term 1 but all other terms 0.*

Prove that for any $\mathbf{x} \in T$, $\mathbf{x} \mapsto 0$.

Because 2^i and 3^i are relatively prime, by Bezout's lemma, x_i can be written in the form $a_i 2^i + b_i 3^i$. Then,

$$\begin{aligned} f(\mathbf{x}) &= f((a_1 2^1 + b_1 3^1, a_2 2^2 + b_2 3^2, a_3 2^3 + b_3 3^3, \ldots)) \\ &= f((a_1 2^1, a_2 2^2, a_3 2^3, \ldots)) + f((b_1 3^1, b_2 3^2, b_3 3^3, \ldots)) \\ &= 0 + 0 = 0, \end{aligned}$$

as desired!

Problem of my own creation! (2022 ELMO 3)

[1] meaning the number such that 50% of the time, the distance is less.

Problem of my own creation! (2022 ELMO 3)

Problem

Given two uniformly random points on the perimeter of a regular 2022-gon of unit area,

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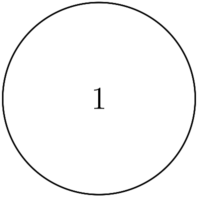

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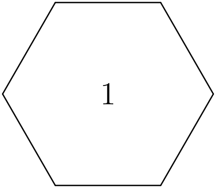

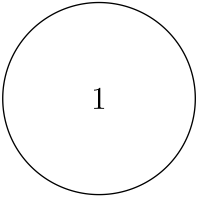

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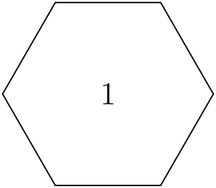

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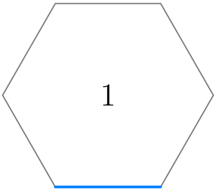

- Assume A is on the bottom

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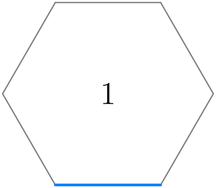

- Assume A is on the bottom
- B has an equal chance of being on top or on bottom, so those cancel out

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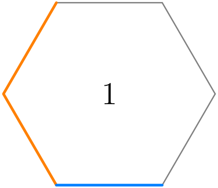

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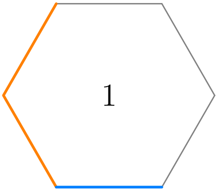

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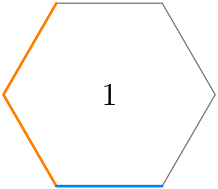

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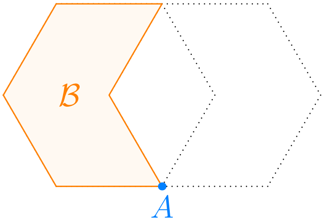

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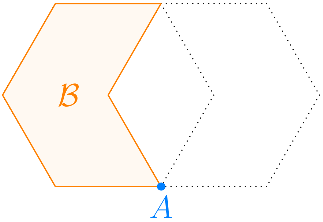

- B is uniformly distributed across region $\mathcal{B}$

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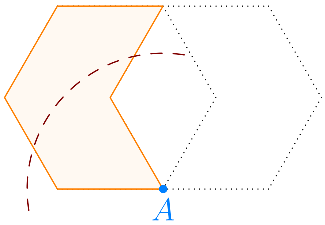

- B is uniformly distributed across region $\mathcal{B}$
- Circle bisects region $\mathcal{B}$

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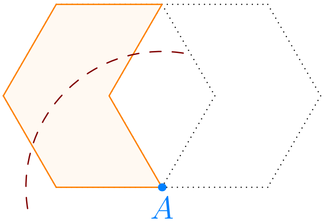

- B is uniformly distributed across region $\mathcal{B}$
- Circle bisects region $\mathcal{B}$
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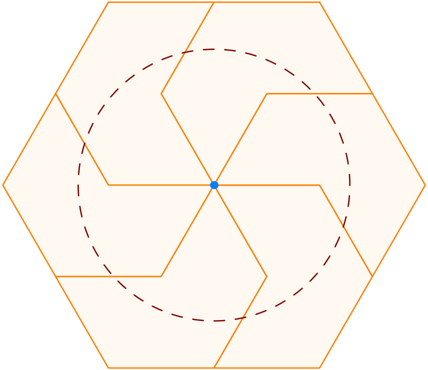

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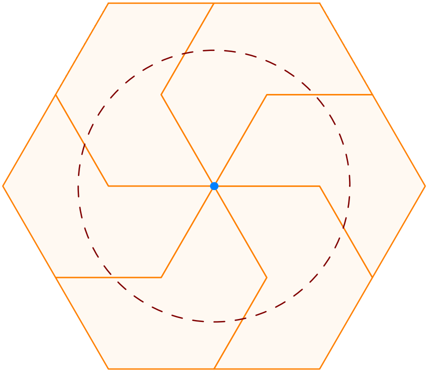

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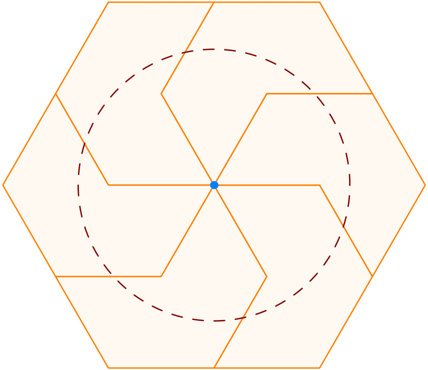

- B is uniformly distributed across region $\mathcal{B}$
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- Circle has area 2

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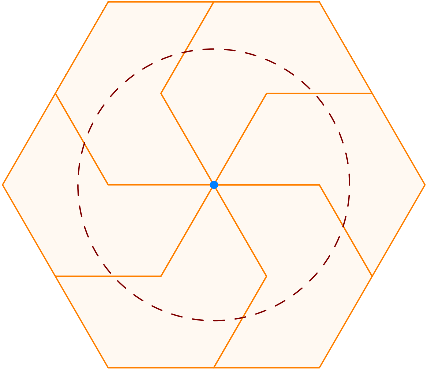

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- $\pi r^2 = 2,\ r = \boxed{\sqrt{\frac{2}{\pi}}}$

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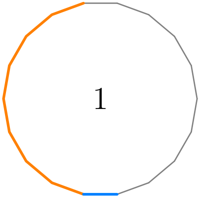

- Assume A is on the bottom
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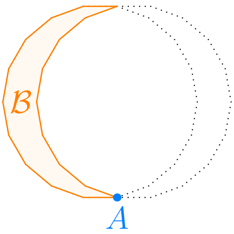

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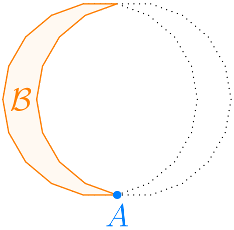

- Issue: B isn't uniformly distributed across region $\mathcal{B}$

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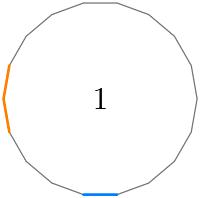

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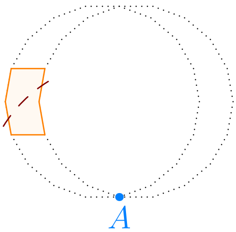

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- Circle bisects region $\mathcal{B}$
- Inner half was supposed to look like a puzzle piece...

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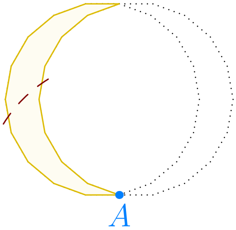

- B is uniformly distributed across region $\mathcal{B}$
- Circle bisects region $\mathcal{B}$
- Inner half was supposed to look like a puzzle piece...
- Add the same to both sides!

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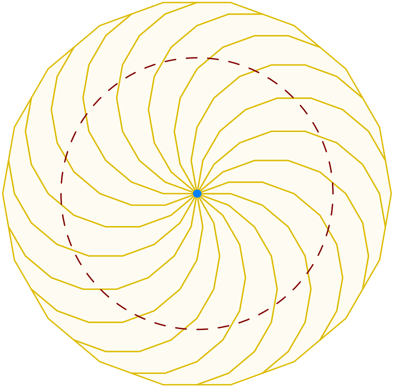

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- Inner half looks like a puzzle piece...
- Add the same to both sides!
- 2022-gon has area 4
- Circle has area 2
- $\pi r^2 = 2$, $r = \boxed{\sqrt{\frac{2}{\pi}}}$

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Thank you!

Magic Math

Espen Slettnes

May 7, 2023

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