Linear Algebra in Combinatorics

Espen Slettnes

Berkeley Math Circle
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Exercise 1. Let P be a nonzero polynomial. Show that P has a nonzero multiple where the degree of every term is prime. For instance, if $P(x) = x^2 - x + 5$, such a multiple would be $x^5 + 4x^3 + 5x^2$.

Walkthrough:

(a) Interpret the multiples of P with linear algebra.

(b) Conclude with a dimensionality argument.

Exercise 2. (Lindström) Prove that for any family $A_1, \ldots, A_m$ of $m \geq n+1$ subsets of $[n]$ there are disjoint $I_1, I_2 \subset I$ such that $I_1 \cup I_2 \neq \varnothing$ and so that $\bigcup_{i \in I_1} A_i = \bigcup_{j \in I_2} A_j$.

Walkthrough:

(a) Define m relevant vectors.

(b) Conclude by their linear dependence.

Exercise 3. (Lindström) Prove that for any family $A_1, \ldots, A_m$ of $m \geq n+2$ subsets of $[n]$ there are disjoint $I_1, I_2 \subset I$ such that $I_1 \cup I_2 \neq \varnothing$ and so that $\bigcup_{i \in I_1} A_i = \bigcup_{j \in I_2} A_j$ and $\bigcap_{i \in I_1} A_i = \bigcap_{j \in I_2} A_j$.

Walkthrough:

(a) Define a better set of m relevant vectors.

(b) Show that they are dependent.

(c) Conclude.

Exercise 4. (Helly) Show that if more than n convex sets in n dimensions are such that any $n+1$ of them intersect, they all intersect.

Walkthrough:

(a) Show that any $n+2$ vectors in $\mathbb{R}^n$ are affine dependent, meaning there is a linear combination of them that is zero whose sum of coefficients is zero.

(b) Prove Radon's theorem, which states that any set of $n+2$ points in $\mathbb{R}^n$ can be partitioned into two sets with intersecting convex hulls.

(c) For $n+2$ points $P_1, \ldots, P_{n+2}$ and $n+2$ convex sets $S_1, \ldots, S_{n+2}$, and given $P_i \in S_j$ if $i \neq j$, show all S's intersect.

(d) Conclude by induction.

Exercise 5. (Dehn) Show that a cube cannot be cut with a straight knife and reassembled into a regular tetrahedron.

Walkthrough:

(a) Choose a field in which to work.

(b) Show that the dihedral angle of a tetrahedron is $\alpha = \arccos(1/3)$. If you are adventurous, show that α/π is irrational.

(c) Assuming the axiom of choice, show that there is a linear transformation $T : \mathbb{R} \to \mathbb{R}$ where $T(\pi) = 0$ and $T(\alpha) = 1$.
(d) For an edge e, let $\ell(e)$ be its length and $\theta(e)$ be its dihedral angle. Let $D(P) = \sum_{e \in P} \ell(e)T(\theta(e))$. Show that for Q a cube and S a regular tetrahedron of equal volume, $D(Q) \neq D(S)$.
(e) Show that if P can be cut into P_1 and P_2, $D(P) = D(P_1) + D(P_2)$.
(f) Assuming the axiom of choice, show that a cube cannot be sliced and reassembled into a regular tetrahedron.
(g) Conclude by modifying the above argument to not rely on the axiom of choice.

Exercise 6. (Sutner) There is a network of lights which are initially off. Whenever you toggle the state of any light, the state of all the directly connected lights are toggled as well. Show that you can get all lights to be on concurrently.

Walkthrough: (Proof due to Caro)

(a) Choose a field in which to work.
(b) Define a relevant $n \times n$ matrix A and a vector $\mathbf{y}$ such that it suffices to show $\mathbf{y} \in \operatorname{Im} A$.
(c) Show that if you have an odd number of lights, flipping all switches must necessarily leave at least one light on.
(d) Show that after flipping an odd number of switches, at least one light must be on.
(e) Show that A does not have an odd number of rows summing to zero.
(f) Show that $\mathbf{y} \in (\ker A)^{\perp}$.
(g) Conclude.

Exercise 7. (Berlekamp) Oddtown has a population of n, and many clubs, each with an odd number of members. (No two clubs have the same membership roster). Whenever two clubs compete with each other, they split their common members evenly between them, so any two clubs in Oddtown share an even number of members. What is the maximum number of clubs that can exist in Oddtown?

Walkthrough:

(a) Choose a field in which to work.
(b) Define a relevant $m \times n$ matrix A where m is the number of clubs.
(c) What is AA^T?
(d) Conclude by looking at the rank.

Exercise 8. (Berlekamp) Eventown has a population of n, and many clubs, each with an even number of members. As in Oddtown, any two clubs in Eventown share an even number of members. What is the maximum number of clubs that can exist in Eventown?

Walkthrough:

(a) Choose a field in which to work.
(b) Define a relevant $m \times n$ matrix A where m is the number of clubs.
(c) Show that the rowspace of A is a subspace of the kernel of A.
(d) Conclude with the rank-nullity theorem.

Exercise 9. (Nonuniform Fisher Inequality)

Wedville has a population of n, and many clubs. Whenever two people marry, the bride and groom will each want to invite their friends from one of the clubs where s/he is a member; the final invite list will be those that are members of both clubs. Since the only catering service in town can only accommodate k people, any two clubs in Wedville have exactly k members in common. Show that the number of clubs in Wedville is at most n.

Walkthrough:

(a) Define a relevant vector for every club.
(b) Show that if the magnitude of a linear combination of these vectors is zero, the coefficients are zero.
(c) Conclude.

Exercise 10. (Frankl-Wilson) Eyeing an opportunity, some new catering services set up shop in Wedville; there are now s caterers that can accommodate $k_1, k_2, \ldots, k_s$ guests, respectively. This also allows some new clubs to take root; any two clubs of Wedville now have either $k_1, k_2, \ldots,$ or k_s members in common. Show that Wedville now has at most $\sum\limits_{i=0}^{s} \binom{n}{i}$ clubs.

Walkthrough:

(a) Define a relevant vector for every club.
(b) For such a vector $\mathbf{v}$ belonging to a club of size C, let $f_{\mathbf{v}}(\mathbf{x}) := \prod\limits_{k_i < C} (\mathbf{v} \bullet \mathbf{x} - k_i)$.

Show that all such polynomials are linearly independent.
(c) Using that $x^2 = x$ for $x \in \{0, 1\}$, modify them to lie in a space of dimension $\sum\limits_{i=0}^{s} \binom{n}{i}$.
(d) Conclude.

Exercise 11. (Graham-Pollak) Doublen has a population of n, and exactly two of each available service: two grocery stores, two banks, two barbers, two doctors, and so on. Its residents are extremely loyal: each person frequents at most one of each service. Moreover, they are unique, but agreeable: any two people frequent *exactly* one competing set of services; e.g., Anna and Belle could visit different doctors, but not go to different banks or shop at different grocery stores. Show that there exist at least $(n-1)$ types of services in Doublen.

Walkthrough:

(a) Show that the rank of a sum of matrices is at most the sum of the ranks of the matrices.
(b) Define a relevant matrix of rank 1 for every type of service.
(c) Show that the sum S of all these matrices satisfies $S + S^T = J - I$, where J is the $n \times n$ matrix of 1's and I is the $n \times n$ identity matrix.
(d) Show that if $J\mathbf{z} = S\mathbf{z} = 0$, $\mathbf{z} = 0$.
(e) Conclude.

Exercise 12. (Independently: Perles-Shelah, Sauer, and Vapnik-Chervonenkis)
Swaytown has a population of n and a number of different clubs. Any k townspeople can be divided into leaders and followers, such that if all leaders belong to a certain club, at least one follower does as well. Show that the number of clubs in Swaytown is at most $\sum\limits_{i=0}^{k-1} \binom{n}{i}$.

Walkthrough:

(a) Let $S_1, \ldots, S_r$ be the clubs. Assume for the sake of contradiction that $r > \sum\limits_{i=0}^{k-1} \binom{n}{i}$. Show that there are real numbers $a_1, \ldots a_r$ such that $f(S) = \sum\limits_{S \subseteq S_i} a_i$ is 0 if $|S| < k$.
(b) Show that there is a set S such that $f(S) \neq 0$.
(c) Let S be minimal. Show that for any subset $A \subseteq S$, $\sum\limits_{S_i \cap S = A} a_i \neq 0$.
(d) Conclude.

Exercise 13. (A special case of Bollobás's Theorem). Let $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be distinct finite sets of size a and b, respectively, such that $A_i \cap B_i = \emptyset$ for all i, but $A_i \cap B_j \neq \emptyset$ for all $i \neq j$. Prove that $n \leq \binom{a+b}{a}$.

(Bollobás's Theorem was also on my Probabilistic Method handout from fall.)

Walkthrough: (Proof due to Frankl)

(a) Let X be the union of all A_i and B_j. Take a set $V \subset \mathbb{R}^{a+1}$ of $|X|$ vectors in general position. Define subsets A'_i and B'_j of V.

(b) Show that there exists a vector $\mathbf{a}_i$ such that for $\mathbf{v} \in V$, $\mathbf{v} \bullet \mathbf{a}_i = 0$ iff $\mathbf{v} \in A_i$.

(c) Let $f_j(\mathbf{x}) := \prod_{\mathbf{v} \in B'_j} \mathbf{v} \bullet \mathbf{x}$. Evaluate $f_j(\mathbf{a}_i)$ and show that the f_j's are linearly independent.

(d) Find a space of dimension $\binom{a+b}{a}$ that contains every f_j.

(e) Conclude.

Exercise 14. (Blokhuis) A two-distance set is a set of points such that there are two positive numbers d_1 and d_2 where the distance between any two points in the set is either d_1 or d_2. Show that the size of a two-distance set in n dimensions is at most $\binom{n+2}{2}$.

Walkthrough: (Proof due to Frankl and Pach)

(a) Let S be a two-distance set in n dimensions. For $\mathbf{a} \in S$, let $f_{\mathbf{a}}(\mathbf{x}) := \left(\|\mathbf{x}-\mathbf{a}\|^2 - d_1^2\right)\left(\|\mathbf{x}-\mathbf{a}\|^2 - d_2^2\right)$.
Evaluate $f_{\mathbf{a}}(\mathbf{b})$ for $\mathbf{a}, \mathbf{b} \in S$.

(b) Show that $f_{\mathbf{a}}(\mathbf{x})$ lies in the span of $\|x\|^4,\ \|x\|^2 x_i,\ x_i x_j,\ x_i$, and 1.

(c) What are the coefficients of t^4 and t^3 in $f_{\mathbf{a}}(t\mathbf{e}_i)$?

(d) Show that the $f_{\mathbf{a}}$'s span does not contain any linear functions.

(e) Conclude.

Exercise 15. (Hoffman-Singleton) A graph has girth 5 if its minimum cycle has length 5, and is d-regular if all its vertices have degree d. For which d's do there exist a graph of girth 5 on d^2+1 vertices?

Walkthrough:

(a) Let A be the adjacency matrix of such a graph. Show that $A^2 + A - (d-1)I = J$, where J is the $(d^2+1) \times (d^2+1)$ matrix of 1's and I is the $(d^2+1) \times (d^2+1)$ identity matrix.

(b) Prove the principal axis theorem, which states that real symmetric matrices have orthogonal eigenbases.

(c) Find an eigenvector of A with eigenvalue d.

(d) Show that for all other eigenvectors $\mathbf{v}$ of A, $J\mathbf{v} = 0$.

(e) Show that all other eigenvalues λ are of the form $\lambda = (-1 \pm s)/2$, where $s = \sqrt{4d-3}$.

(f) Let $\lambda_+ = (-1+s)/2$ and $\lambda_- = (-1-s)/2$, and let m_+ and m_- be the multiplicities of the eigenvalues and λ_+ and λ_-, respectively. Show that $d + m_+\lambda_+ + m_-\lambda_- = 0$.

(g) Show that $2d - d^2 + (m_+ - m_-)s = 0$, and conclude that s is an integer or $d = 2$.

(h) Write the previous equation as a quartic in s, and show that if $d \neq 2$, $s \mid 15$

(i) Find a graph for $d = 2$ and for $s = 3$.
The Hoffman-Singleton graph gives $s = 5$. It is an open question whether there is a graph with $s = 15$.