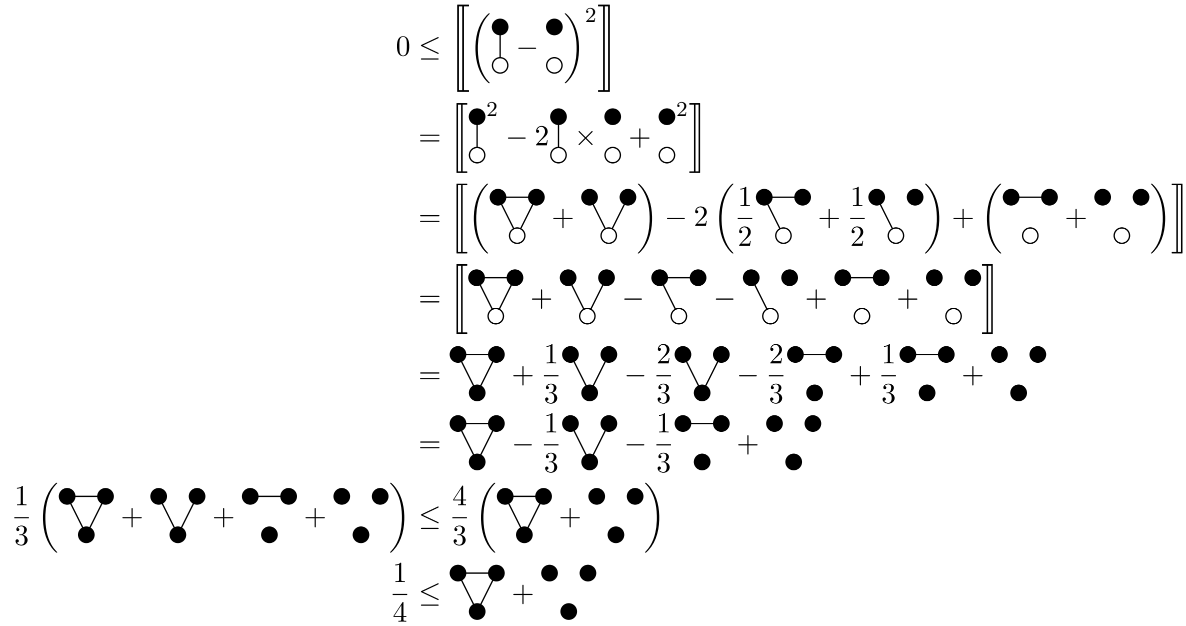
Flag Algbras

An informal introduction

Espen Slettnes

Today, our goal is to be able to understand the following:



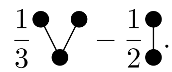
Let's get started!

Unlabeled Flags

Definition 1. Given a subset of vertices $S \subseteq V(G)$ in graph G, the subgraph *induced* in G by S is the graph with vertex set S and edge set $\{\{u,v\} \mid u,v \in S, \{u,v\} \in G\}$.

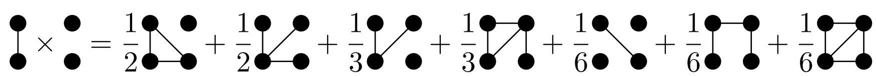
Definition 2. The *density* of graph F in graph G, denoted $p(F;G)$, is the probability that the subgraph induced in G by $|V(F)|$ random vertices is F.
We also denote by $p(F_1,F_2;G)$ the probability that $|V(F_1)|$ random vertices and $|V(F_2)|$ random remaining vertices in G induce F_1 and F_2, respectively.

Unlabeled flags are graphs equipped with the following operations:

- $\bullet = 1$.
- Vector space operations and axioms, e.g., we can write .
- We can multiply flags, as follows:

$$F_1 \times F_2 = \sum_{|V(F)|=|V(F_1)|+|V(F_2)|} p(F_1,F_2;F)F$$

For example,



A flag F when used in equations represents $p(F;G)$ for a very large graph G (more formally, a sequence of graphs of increasing size, where $p(F;G)$ converges for all F).

Exercise 3. What is ?

Exercise 4. Explain why

$$p(F_1;G) \times p(F_2;G) \approx \sum_{|V(F)|=|V(F_1)|+|V(F_2)|} p(F_1,F_2;F)p(F;G)$$

in a very large graph G.

Exercise 5. Using flag operations, show that

$$\sum_{|V(F)|=n} F = 1$$

and explain why this makes sense given the interpretation of flags above. *Hint: Look at* $\bullet^n$.

Exercise 6. Using flag operations, show that

$$F = \sum_{|V(F')|=n} p(F;F')F$$

and explain why this makes sense given the interpretation of flags above. *Hint: Look at* $\bullet^m \times F$.

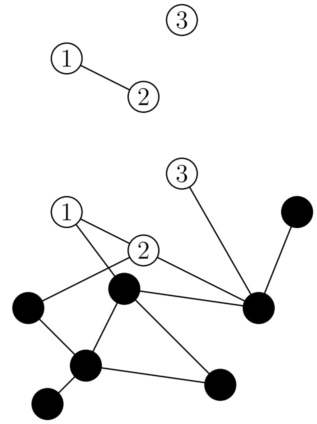
Exercise 7. Extend p to $p(F_1,\dots,F_m;G)$ so that

$$\prod_{i=1}^{m} F_i = \sum_{|V(F)|=\sum_{i=1}^{m}|V(F_i)|} p(F_1,\dots,F_m;F)F.$$

Also conclude that flag multiplication is associative.

σ-Flags

For a labeled graph σ, a *σ-graph* is a graph with some labeled vertices inducing σ and otherwise unlabeled vertices. We do not include the labeled vertices in the vertex set, nor the edges between them (edges from labeled to unlabeled vertices are included, however).



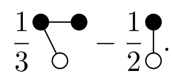
A labeled graph σ (top) and a σ-graph (bottom).

Definition 8. The *density* of a σ-graph F_σ in a σ-graph G_σ, denoted $p_\sigma(F_\sigma; G_\sigma)$, is the probability that the σ-subgraph induced in G_σ by the labeled vertices and $|V(F_\sigma)|$ random vertices is F_σ.

We also denote by $p_\sigma(F_{1\sigma}, F_{2\sigma}; G_\sigma)$ the probability that σ along with both $|V(F_{1\sigma})|$ random vertices and $|V(F_{2\sigma})|$ random remaining vertices in G_σ induce $F_{1\sigma}$ and $F_{2\sigma}$, respectively.

σ-flags are σ-graphs equipped with the following operations:

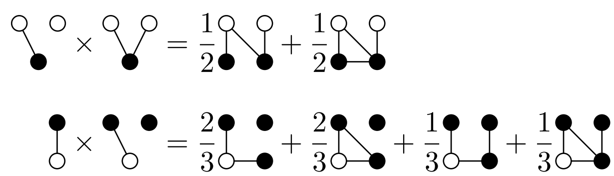
– $\sum_{|V(F_\sigma)|=1} F_\sigma = 1.$

– Vector space operations and axioms, e.g., we can write 

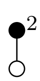
– We can multiply σ-flags, as follows:

$$F_{1\sigma} \times F_{2\sigma} = \sum_{|V(F_\sigma)|=|V(F_{1\sigma})|+|V(F_{2\sigma})|} p_\sigma(F_{1\sigma}, F_{2\sigma}; F_\sigma) F_\sigma$$

For example,



A σ-flag F_σ when used in equations represents $p_\sigma(F_\sigma; G_\sigma)$ for a very large σ-graph G_σ (more formally, a sequence of σ-graphs of increasing size, where $p_\sigma(F_\sigma; G_\sigma)$ converges for all F_σ).

Exercise 9. What is  ?

Exercise 10. Explain why

$$p_\sigma(F_{1\sigma}; G_\sigma) \times p_\sigma(F_{2\sigma}; G_\sigma) \approx \sum_{|V(F_\sigma)|=|V(F_{1\sigma})|+|V(F_{2\sigma})|} p_\sigma(F_{1\sigma}, F_{2\sigma}; F_\sigma) p_\sigma(F_\sigma; G_\sigma)$$

in a very large graph G_σ.

Exercise 11. Using σ-flag operations, show that

$$\sum_{|V(F_\sigma)|=n} F_\sigma = 1$$

and explain why this makes sense given the interpretation of σ-flags above. *Hint: Look at* $\left(\sum_{|V(F_\sigma)|=1} F_\sigma\right)^n$.

Exercise 12. Using σ-flag operations, show that

$$F_\sigma = \sum_{|V(F'_\sigma)|=n} p_\sigma(F_\sigma; F'_\sigma) F_\sigma$$

and explain why this makes sense given the interpretation of σ-flags above. *Hint: Look at* $\left(\sum_{|V(F_\sigma)|=1} F_\sigma \right)^m \times F_\sigma$.

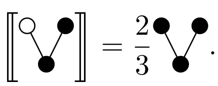
Exercise 13. Extend p_σ to $p_\sigma(F_{1\sigma}, \ldots, F_{m\sigma}; G_\sigma)$ so that

$$\prod_{i=1}^{m} F_{i\sigma} = \sum_{|V(F_\sigma)|=\sum_{i=1}^{m} |V(F_{i\sigma})|} p_\sigma(F_{1\sigma}, \ldots, F_{m\sigma}; F_\sigma) F_\sigma.$$

Also conclude that σ-flag multiplication is associative.

The Averaging Operator

Definition 14. The *averaging operator* (a.k.a. *downward operator*) on a σ-flag F_σ, denoted $[\![F_\sigma]\!]$, is PF, where F is the unlabeled flag formed by removing the labels from F_σ and P and is the probability that putting $|V(\sigma)|$ labels randomly on F gives F_σ.
Additionally, we make the averaging operator a linear operator.

For example, 

An interpretation of the averaging operator is that $[\![F_\sigma]\!]$ is the probability that labeling $|V(\sigma)|$ vertices of a very large graph G_σ then choosing $|F_\sigma|$ vertices induces F_σ.

Theorem 15. *(Razborov) For any two linear combinations of flags $\mathcal{F}_1$ and $\mathcal{F}_2$,*

$$[\![\mathcal{F}_{1\sigma}^2]\!] \times [\![\mathcal{F}_{2\sigma}^2]\!] \geq [\![\mathcal{F}_{1\sigma}\mathcal{F}_{2\sigma}]\!]^2$$

Exercise 16. Show that $[\![\mathcal{F}_\sigma^2]\!] \geq 0$ for all linear combinations of flags.

Now let's look at the equation on page 1.

Exercise 17. What have we proved? Make sure you are comfortable with this proof.

Exercise 18. Show that the number of triangles in a graph G is at least $(1 - o(1)) \cdot \left(\frac{4|E(G)|^2}{3|V(G)|} - \frac{|E(G)||V(G)|}{3} \right)$.
Here, $1 - o(1)$ denotes a value that gets arbitrarily close to 1 as G grows large.
Hint: Expand: