An Assorted Collection of Problems that Deserve to be in a Lecture but that are Individually Too Short

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1 Kickoff

Beaut 1. Show that the lines connecting the midpoints of a triangle's respective bases and altitudes concur.

2 Posers

Beaut 2. (2016 Olympic Revenge 5) Let T the set of infinite sequences of integers. For any two elements $(a_1, a_2, a_3, ...)$ and $(b_1, b_2, b_3, ...)$ in T, define the sum

$$(a_1, a_2, a_3, ...) + (b_1, b_2, b_3, ...) := (a_1 + b_1, a_2 + b_2, a_3 + b_3, ...).$$

Let $f : T \to \mathbb{Z}$ a function such that:

i) If $x \in T$ has exactly one term equal to 1 and all others equal to 0, then $f(x) = 0$.

ii) $f(x + y) = f(x) + f(y)$, for all $x, y \in T$.

Prove that $f(x) = 0$ for all $x \in T$.

Beaut 3 (Graham-Pollak)**.** What is the minimum size of a partition of K_n into complete bipartite graphs?

3 CAMO 2022

Beaut 4 (CAMO 2)**.** Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(f(x)f(y)) = f(xf(y)) + f(y)$$

for all integers x and y.

Beaut 5 (CAMO 3)**.** Let $a_1 < a_2 < \cdots < a_n$ be positive integers such that the set of positive integers can be partitioned into an infinite number of sets, each of the form $\{a_1k, a_2k, \ldots, a_nk\}$ for some positive integer k. Prove that $a_i \mid a_n$ for all $1 \le i \le n$.

Beaut 6 (CAMO 6)**.** Let $A_1A_2A_3A_4A_5$ be a convex pentagon satisfying $\overline{A_{i-1}A_{i+1}} \parallel \overline{A_{i-2}A_{i+2}}$ for all i, where all indices are considered modulo 5. Prove that there exist points B_1, B_2, B_3, B_4, B_5 in the plane such that

- B_i lies on line $A_{i-2}A_{i+2}$,
- the five lengths A_iB_i are equal, and
- the five lines A_iB_i are concurrent.